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# WHITTAKER FUNCTIONS ON GROUPS OF LOW RANK

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§1. We shall discuss two topics in this report. One is the commutation relations among differential operators. The other is concerned with an explicit formula of Whittaker functions on $Sp_2(\mathbf{R})$. Whittaker functions on other groups of low rank are rather known. See [2],[4],[16],[17] for instance.

As usual, we consider an element in the center of the universal enveloping algebra of Lie algebra of Lie groups $G$ as a differential operator on $G$. Generators of the center of the universal enveloping algebra of $\mathfrak{sp}(2,\mathbf{R})$ are given in [6] and the way to find generators of that of Lie algebra of classical groups in [6],[3].

Put

$$H_1=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix},\quad H_2=\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix},$$

$$X_1=\begin{pmatrix}0&1&0&0\\0&0&0&0\\0&0&0&0\\0&0&-1&0\end{pmatrix},\quad X_2=\begin{pmatrix}0&0&0&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{pmatrix},$$

$$X_3=\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\quad X_4=\begin{pmatrix}0&0&1&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}.$$

Then the generators of the center of the universal enveloping algebra of $\mathfrak{sp}(2,\mathbf{R})$ in [6] are

$$\begin{aligned}\lambda(L_1)=&H_1H_1+H_2H_2+6H_1\\&+2H_2+4X_{-1}X_1+8X_{-4}X_4+4X_{-3}X_3+8X_{-2}X_2,\end{aligned}$$

$$\begin{aligned}
\lambda(L_2) = & 16X_{-4}X_{-4}X_4X_4 + 16X_{-4}X_{-3}X_3X_4 \\
& - 32X_{-4}X_{-2}X_2X_4 + 16X_{-4}X_{-2}X_3X_3 \\
& + 16X_{-4}X_{-1}X_1X_4 + 8X_{-4}H_1H_2X_4 \\
& + 8X_{-4}(H_1 - H_2)X_1X_3 - 16X_{-4}X_1X_1X_2 \\
& + 16X_{-3}X_{-3}X_2X_4 + 16X_{-3}X_{-2}X_2X_3 \\
& + 8X_{-3}X_{-1}(H_1 - H_2)X_4 + 4X_{-3}H_2H_2X_3 \\
& + 8X_{-3}(H_1 + H_2)X_1X_2 + 16X_{-2}X_{-2}X_2X_2 \\
& - 16X_{-2}X_{-1}X_{-1}X_4 + 8X_{-2}X_{-1}(H_1 + H_2)X_3 \\
& + 16X_{-2}X_{-1}X_1X_2 - 8X_{-2}H_1H_2X_2 \\
& + 4X_{-1}H_1H_1X_1 + H_1H_1H_2H_2 \\
& - 16X_{-4}H_1X_4 + 32X_{-4}H_2X_4 + 32X_{-4}X_1X_3 \\
& + 32X_{-3}X_{-1}X_4 - 8X_{-3}H_1X_3 + 16X_{-3}X_1X_2 \\
& + 16X_{-2}X_{-1}X_3 - 16X_{-2}(H_1 + H_2)X_2 \\
& + 24X_{-1}H_1X_1 + 2H_1H_1H_2 \\
& + 6H_1H_2H_2 \\
& - 48X_{-4}X_4 - 24X_{-3}X_3 - 48X_{-2}X_2 \\
& + 24X_{-1}X_1 - 2H_1H_1 + 12H_1H_2 \\
& + 6H_2H_2 - 12H_1 + 12H_2.
\end{aligned}$$

In order to discribe generators of the center of the universal enveloping algebra of $\mathfrak{sl}(2, \mathbf{R})$, put

$$B_{12} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B_{13} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$B_{14} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B_{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$B_{24} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B_{34} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$B_{11} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B_{22} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$B_{33} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad B_{ij} = {}^tB_{ji} \quad (i > j),$$

and define a symmetrizer $S_n$ on the space of differential operators by

$$S_n(X_1, X_2, \ldots, X_n) = \sum_{\sigma \in \mathfrak{S}_n} X_{\sigma(1)} X_{\sigma(2)} \cdots X_{\sigma(n)}.$$

Then the generators are

$$\begin{aligned} \beta_2 = -\,\{&3S_2(B_{11}, B_{11}) + 4S_2(B_{11}, B_{22}) + 2S_2(B_{11}, B_{33}) \\ &+ 8S_2(B_{12}, B_{21}) + 8S_2(B_{13}, B_{31}) + 8S_2(B_{14}, B_{41}) \\ &+ 4S_2(B_{22}, B_{22}) + 4S_2(B_{22}, B_{33}) + 8S_2(B_{23}, B_{32}) \\ &+ 8S_2(B_{24}, B_{42}) + 3S_2(B_{33}, B_{33}) + 8S_2(B_{34}, B_{43})\}/128, \end{aligned}$$

$$\begin{aligned} \beta_3 = -\,\{&S_3(B_{11}, B_{11}, B_{11}) + 2S_3(B_{11}, B_{11}, B_{22}) \\ &+ S_3(B_{11}, B_{11}, B_{33}) + 4S_3(B_{11}, B_{12}, B_{21}) \\ &+ 4S_3(B_{11}, B_{13}, B_{31}) + 4S_3(B_{11}, B_{14}, B_{41}) \\ &- 4S_3(B_{11}, B_{23}, B_{32}) - 4S_3(B_{11}, B_{24}, B_{42}) \\ &- S_3(B_{11}, B_{33}, B_{33}) - 4S_3(B_{11}, B_{34}, B_{43}) \\ &+ 8S_3(B_{12}, B_{21}, B_{22}) + 4S_3(B_{12}, B_{21}, B_{33}) \\ &+ 8S_3(B_{12}, B_{23}, B_{31}) + 8S_3(B_{12}, B_{24}, B_{41}) \\ &+ 8S_3(B_{13}, B_{21}, B_{32}) + 4S_3(B_{13}, B_{31}, B_{33}) \\ &+ 8S_3(B_{13}, B_{34}, B_{41}) + 8S_3(B_{14}, B_{21}, B_{42}) \\ &+ 8S_3(B_{14}, B_{31}, B_{43}) - 4S_3(B_{14}, B_{33}, B_{41}) \end{aligned}$$

$$
\begin{aligned}
&- 2S_3(B_{22},B_{33},B_{33}) - 8S_3(B_{22},B_{34},B_{43})\\
&+ 4S_3(B_{23},B_{32},B_{33}) + 8S_3(B_{23},B_{34},B_{42})\\
&+ 8S_3(B_{24},B_{32},B_{43}) - 4S_3(B_{24},B_{33},B_{42})\\
&- S_3(B_{33},B_{33},B_{33}) - 4S_3(B_{33},B_{34},B_{43})\}/512,
\end{aligned}
$$

$$
\begin{aligned}
\beta_4 = &- (3S_4(B_{11},B_{11},B_{11},B_{11}) + 8S_4(B_{11},B_{11},B_{11},B_{22})\\
&+ 4S_4(B_{11},B_{11},B_{11},B_{33}) + 16S_4(B_{11},B_{11},B_{12},B_{21})\\
&+ 16S_4(B_{11},B_{11},B_{13},B_{31}) + 16S_4(B_{11},B_{11},B_{14},B_{41})\\
&- 8S_4(B_{11},B_{11},B_{22},B_{22}) - 8S_4(B_{11},B_{11},B_{22},B_{33})\\
&- 48S_4(B_{11},B_{11},B_{23},B_{32}) - 48S_4(B_{11},B_{11},B_{24},B_{42})\\
&- 14S_4(B_{11},B_{11},B_{33},B_{33}) - 48S_4(B_{11},B_{11},B_{34},B_{43})\\
&+ 64S_4(B_{11},B_{12},B_{21},B_{22}) + 32S_4(B_{11},B_{12},B_{21},B_{33})\\
&+ 64S_4(B_{11},B_{12},B_{23},B_{31}) + 64S_4(B_{11},B_{12},B_{24},B_{41})\\
&+ 64S_4(B_{11},B_{13},B_{21},B_{32}) + 32S_4(B_{11},B_{13},B_{31},B_{33})\\
&+ 64S_4(B_{11},B_{13},B_{34},B_{41}) + 64S_4(B_{11},B_{14},B_{21},B_{42})\\
&+ 64S_4(B_{11},B_{14},B_{31},B_{43}) - 32S_4(B_{11},B_{14},B_{33},B_{41})\\
&- 32S_4(B_{11},B_{22},B_{22},B_{22}) - 48S_4(B_{11},B_{22},B_{22},B_{33})\\
&- 128S_4(B_{11},B_{22},B_{23},B_{32}) - 128S_4(B_{11},B_{22},B_{24},B_{42})\\
&- 8S_4(B_{11},B_{22},B_{33},B_{33}) + 64S_4(B_{11},B_{22},B_{34},B_{43})\\
&- 160S_4(B_{11},B_{23},B_{32},B_{33}) - 192S_4(B_{11},B_{23},B_{34},B_{42})\\
&- 192S_4(B_{11},B_{24},B_{32},B_{43}) + 32S_4(B_{11},B_{24},B_{33},B_{42})\\
&+ 4S_4(B_{11},B_{33},B_{33},B_{33}) + 32S_4(B_{11},B_{33},B_{34},B_{43})\\
&+ 64S_4(B_{12},B_{21},B_{22},B_{22}) + 64S_4(B_{12},B_{21},B_{22},B_{33})\\
&- 48S_4(B_{12},B_{21},B_{33},B_{33}) - 256S_4(B_{12},B_{21},B_{34},B_{43})\\
&+ 128S_4(B_{12},B_{22},B_{23},B_{31}) + 128S_4(B_{12},B_{22},B_{24},B_{41})\\
&+ 192S_4(B_{12},B_{23},B_{31},B_{33}) + 256S_4(B_{12},B_{23},B_{34},B_{41})\\
&+ 256S_4(B_{12},B_{24},B_{31},B_{43}) - 64S_4(B_{12},B_{24},B_{33},B_{41})\\
&+ 128S_4(B_{13},B_{21},B_{22},B_{32}) + 192S_4(B_{13},B_{21},B_{32},B_{33})\\
&+ 256S_4(B_{13},B_{21},B_{34},B_{42}) - 64S_4(B_{13},B_{22},B_{22},B_{31})\\
&- 128S_4(B_{13},B_{22},B_{31},B_{33}) - 128S_4(B_{13},B_{22},B_{34},B_{41})
\end{aligned}
$$

$$
\begin{aligned}
&- 256S_4(B_{13}, B_{24}, B_{31}, B_{42}) + 256S_4(B_{13}, B_{24}, B_{32}, B_{41}) \\
&- 48S_4(B_{13}, B_{31}, B_{33}, B_{33}) - 64S_4(B_{13}, B_{33}, B_{34}, B_{41}) \\
&+ 128S_4(B_{14}, B_{21}, B_{22}, B_{42}) + 256S_4(B_{14}, B_{21}, B_{32}, B_{43}) \\
&- 64S_4(B_{14}, B_{21}, B_{33}, B_{42}) - 64S_4(B_{14}, B_{22}, B_{22}, B_{41}) \\
&- 128S_4(B_{14}, B_{22}, B_{31}, B_{43}) + 256S_4(B_{14}, B_{23}, B_{31}, B_{42}) \\
&- 256S_4(B_{14}, B_{23}, B_{32}, B_{41}) - 64S_4(B_{14}, B_{31}, B_{33}, B_{43}) \\
&+ 16S_4(B_{14}, B_{33}, B_{33}, B_{41}) - 16S_4(B_{22}, B_{22}, B_{22}, B_{22}) \\
&- 32S_4(B_{22}, B_{22}, B_{22}, B_{33}) - 64S_4(B_{22}, B_{22}, B_{23}, B_{32}) \\
&- 64S_4(B_{22}, B_{22}, B_{24}, B_{42}) - 8S_4(B_{22}, B_{22}, B_{33}, B_{33}) \\
&+ 64S_4(B_{22}, B_{22}, B_{34}, B_{43}) - 128S_4(B_{22}, B_{23}, B_{32}, B_{33}) \\
&- 128S_4(B_{22}, B_{23}, B_{34}, B_{42}) - 128S_4(B_{22}, B_{24}, B_{32}, B_{43}) \\
&+ 8S_4(B_{22}, B_{33}, B_{33}, B_{33}) + 64S_4(B_{22}, B_{33}, B_{34}, B_{43}) \\
&- 48S_4(B_{23}, B_{32}, B_{33}, B_{33}) - 64S_4(B_{23}, B_{33}, B_{34}, B_{42}) \\
&- 64S_4(B_{24}, B_{32}, B_{33}, B_{43}) + 16S_4(B_{24}, B_{33}, B_{33}, B_{42}) \\
&+ 3S_4(B_{33}, B_{33}, B_{33}, B_{33}) + 16S_4(B_{33}, B_{33}, B_{34}, B_{43}))/65536.
\end{aligned}
$$

§**2.** We define the Weil representation $r_n$ of $Sp_2(\mathbf{R})$ on $V_n = M_{n,2}(\mathbf{R})$ by putting

$$
\begin{aligned}
r_n\begin{pmatrix} E & X \\ 0 & E \end{pmatrix} f\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} &= \exp(2\pi i tr(X{}^tX_1X_2)) f\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \\
r_n\begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix} f\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} &= (\det A)^{n/2} f\begin{pmatrix} X_1A \\ X_2A \end{pmatrix}, \\
r_n\begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix} f\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} &= \int_V\int_V \exp(2\pi i tr({}^tY_1X_2 + {}^tY_2X_1)) \\
&\qquad f\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} dY_1dY_2
\end{aligned}
$$

for $f \in \mathcal{S}(V_n \times V_n), X = {}^tX \in M_{2,2}(\mathbf{R}), A \in M_{2,2}(\mathbf{R})$ with $E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Let $G_1 = SL(2,\mathbf{R}), G_3 = SL(4,\mathbf{R})$. Then we can define representations $\rho_2, \rho_3$ of $G_1 \times G_1, G_3$ on $\mathcal{S}(V_2 \times V_2), \mathcal{S}(V_3 \times V_3)$ in the following manner. First, we define linear mappings $\sigma_1, \sigma_3$ by

$$\sigma_1(X) = \begin{pmatrix} a & d \\ b & -c \end{pmatrix}$$

for

$$X = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \in M_{4,1}(\mathbf{R})$$

and

$$\sigma_3(X) = \begin{pmatrix} 0 & a & b & c \\ -a & 0 & f & -e \\ -b & -f & 0 & d \\ -c & e & -d & 0 \end{pmatrix}$$

for

$$X = \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} \in M_{6,1}(\mathbf{R}).$$

Then $(g,h) \in G_1 \times G_1$ acts on $V_2 \times V_2$ by

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}^{(g,h)} = \left( {\sigma_1}^{-1} \left( {}^t g \left( \sigma_1 \begin{pmatrix} x_{11} \\ x_{21} \\ x_{31} \\ x_{41} \end{pmatrix} \right) h \right), {\sigma_1}^{-1} \left( {}^t g \left( \sigma_1 \begin{pmatrix} x_{12} \\ x_{22} \\ x_{32} \\ x_{42} \end{pmatrix} \right) h \right) \right)$$

for

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \\ x_{41} & x_{42} \end{pmatrix} \in M_{4,2}(\mathbf{R}) = V_2 \times V_2,$$

and $g \in G_3$ acts on $V_3$ by

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}^g$$

$$= \left( \sigma_3{}^{-1} \left( {}^t g \left( \sigma_3 \begin{pmatrix} x_{11} \\ x_{21} \\ x_{31} \\ x_{41} \\ x_{51} \\ x_{61} \end{pmatrix} \right) \right) g \right), \sigma_3{}^{-1} \left( {}^t g \left( \sigma_3 \begin{pmatrix} x_{12} \\ x_{22} \\ x_{32} \\ x_{42} \\ x_{52} \\ x_{62} \end{pmatrix} \right) \right) g \Bigg) \Bigg)$$

for

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \\ x_{41} & x_{42} \\ x_{51} & x_{52} \\ x_{61} & x_{62} \end{pmatrix} \in M_{6,2}(\mathbf{R}) = V_3 \times V_3.$$

Put

$$\rho_2(g)f(X) = f(X^g)$$

for $f \in \mathcal{S}(V_2 \times V_2), g \in G_1 \times G_1$, and put

$$\rho_3(g)f(X) = f(X^g)$$

for $f \in \mathcal{S}(V_3 \times V_3), g \in G_3$. Then, the representations $r_2, r_3, \rho_2, \rho_3$ induce the representations (differential representations) of the center of the universal enveloping algebra of $\mathfrak{sp}(2, \mathbf{R})$, $\mathfrak{sl}(2, \mathbf{R}) \oplus \mathfrak{sp}(2, \mathbf{R})$, $\mathfrak{sl}(4\mathbf{R})$ which we denote by the same letters $r_2, r_3, \rho_2, \rho_3$.

With this notation, we get

THEOREM 1.

$$\begin{aligned}
\rho_3(\beta_2) &= -\frac{1}{32} r_3(\lambda(L_1)), \\
\rho_3(\beta_3) &= 0, \\
\rho_3(\beta_4) &= \frac{3}{512} r_3(\lambda(L_2)) + \frac{1}{128} r_3(\lambda(L_1)), \\
r_2(\lambda(L_1)) &= \rho_2(\gamma, 1) + \rho_2(1, \gamma) - 8, \\
r_2(\lambda(L_2)) &= \rho_2(\gamma, 1)\rho_2(1, \gamma) - 2\rho_2(\gamma, 1) - 2\rho_2(1, \gamma) + 16.
\end{aligned}$$

§3. By using Theorem 1, we can construct Whittaker functions on $Sp_2(\mathbf{R})$ which are standard Whittaker functions not generalized Whittaker functions in [8]. (See [11].) First, we consider same theta functions $\Theta(g, z_1, z_2)$ as in [8] attached to the Weil representation $r_2$ and define a lift

$$F(g) = F_{\varphi_1,\varphi_2}(g) = \int_{\Gamma\backslash H} \Theta(g, z_1, z_2)\varphi_1(z_1)\varphi_2(z_2)d_0z_1d_0z_2$$

where $\varphi_1, \varphi_2$ are Mass wave forms on the upper half plane $H$. Define a character $\Psi_0$ by

$$\Psi_0\left(\begin{pmatrix}1 & n_0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & -n_0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & n_1 & n_2\\ 0 & 1 & n_2 & n_3\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\right) = \exp(2\pi i(n_0 + n_3))$$

as in [10] of the unipotent radical $N$ of a Borel subgroup of $Sp_2(\mathbf{R})$. Considering

$$\int_{N\cap Sp_2(\mathbf{Z})\backslash N} F(ng)\Psi_0(n)dn,$$

we get a following Whittaker function

$$W_{\nu_1,\nu_2}\left(\begin{pmatrix}1 & n_0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & -n_0 & 1\end{pmatrix}\begin{pmatrix}y_1 & 0 & n_1/y_1 & n_2/y_2\\ 0 & y_2 & n_2/y_1 & n_3/y_2\\ 0 & 0 & 1/y_1 & 0\\ 0 & 0 & 0 & 1/y_2\end{pmatrix}k\right)$$
$$= \exp(2\pi i(n_0 + n_3))\int_0^\infty\int_0^\infty v_1^{-1}v_2^{-1}y_1^2y_2K_{\nu_1}(2\pi v_1)K_{\nu_2}(2\pi v_2)$$
$$\exp(-\pi y_1^2/v_1v_2 - \pi v_1v_2/y_2^2 - \pi v_1y_2^2/v_2 - \pi v_2y_2^2/v_1)dv_1dv_2$$

for $k \in SO(4)\cap Sp_2(\mathbf{R})$ with the modified Bessel function $K_\nu$. (See [5].) The latter part of Theorem 1 implies

THEOREM 2.

$$\lambda(L_1)W_{\nu_1,\nu_2} = 4(\lambda_1 + \lambda_2 - 2)W_{\nu_1,\nu_2},$$
$$\lambda(L_2)W_{\nu_1,\nu_2} = 8(2\lambda_1\lambda_2 - \lambda_1 - \lambda_2 + 2)W_{\nu_1,\nu_2}$$

*with* $\lambda_1 = {\nu_1}^2 - 1/4, \lambda_2 = {\nu_2}^2 - 1/4.$

We can calculate Mellin transforms of $W_{\nu_1,\nu_2}$ and can derive an analogie of Barne's second lemma from them by using [9],[10],[15].

## References


1. D. Bump , *Barne's second lemma and its application to Rankin- Selberg convolutions,* Amer. J. of Math. **109** (1987), 179-186.
2. ———, *Automorphic forms of* $GL(3,\mathbf{R})$, Lect. Notes in Math. **1083** (1984).
3. N. Bourbaki, "Éléments de mathématique, Groupes et algèbres de Lie, Chap.7, 8," Hermann.
4. M. Hashizume, *Whittaker functions on semesimple Lie group and their applications,* 数理研講究録 631 (1987) , 123-137.
5. R. Howe and I. I. Piatetski-Shapiro, *Some examples of automorphic forms on* $Sp_4$, Duke Math. J. **50** (1983), 55-106.
6. S. Nakajima , *Invariant differential operators on* $SO(2,q)/SO(2)\times SO(q)$ $(q \geq 3)$, Master these, Univ. of Tokyo.
7. ———, *On invariant differential operators on bounded symmetric domains of type 4*, Proc. Japan Acad. **58, Ser. A** (1982), 235-238.
8. S. Niwa, *On generaliged Whittaker functions on Siegel's uper half space of degree 2*, Nagoya Math. J. **121** (1991), 171-184.
9. M. E. Novodvolsky , *Fonctions J pour* $GSp(4)$, C. R. Acad. Sci. Paris Sér. A **280** (1975), 191-192.
10. ———, *Automorphic L-functions for symplectic group* $GSp(4)$, Proc. Symp. Pure Math. **33** (1979), 87-95.
11. T. Oda, *On Whittaker functions of class 1 on* $Sp(2,\mathbf{R}) = Sp_4(\mathbf{R})$, 数理研講究録 689 (1989) , 148-164.
12. I. I. Piatetski-Shapiro and D. Soudry , *L and* $\epsilon$ *functions for* $GSP(4)\times GL(2)$, Proc. Natl. Acad. Sci. USA **81** (1984), 3924-3927.
13. ———, *Automorphic forms on the symplectic group of order four,* Lecture Notes of I.H.E.S. (1983).
14. D. Soudry , *A uniqueness theorem for representations of* $GSO(6)$ *and the strong multilicity one theorem for generic representations of* $GSp(4)$, Israel J. of Math. **58** (1987), 257-287.
15. ———, *The L and* $\gamma$ *factors for generic representations of* $GSp(4,k)\times GL(2,k)$ *over a local nonarchimedean field k*, Duke Math. J. **51** (1984), 355-394.
16. E. Stade , *On explicit integral formulas for* $GL(n,\mathbf{R})$*-Whittaker functions,* Duke Math. J. **60** (1990), 313-362.
17. ———, *Poincaré series for* $GL(3,\mathbf{R})$*-Whittaker functions,* Duke Math. J. **58** (1989), 695-729.
18. H. Yoshida, *The action of Hecke operators on theta series,* Algebraic and Topological Theories (1985), 197-238.